TMA4122 Mathematics 4M, fall 2013

Solution suggestions

1a Using partial integration we compute

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{\pi} t^2\,\mathrm{d}t = \frac{\pi^2}{6}\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} t^2\cos nt\,\mathrm{d}t\\
&= \frac{1}{\pi}\left[\frac{2t}{n^2}\cos nt + \frac{-2+n^2t^2}{n^3}\sin nt\right]_{t=0}^{\pi}\\
&= \frac{2}{n^2}\cos n\pi = (-1)^n\frac{2}{n^2}\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} t^2\sin nt\,\mathrm{d}t\\
&= \frac{1}{\pi}\left[\frac{2t}{n^2}\sin nt + \frac{2-n^2t^2}{n^3}\cos nt\right]_{t=0}^{\pi}\\
&= \frac{1}{\pi}\left(\frac{2}{n^3}\cos n\pi - \frac{\pi^2}{n}\cos n\pi - \frac{2}{n^3}\right)\\
&= \begin{cases} -\frac{\pi}{n} & \text{if } n \text{ even}\\ \frac{\pi}{n} - \frac{4}{n^3\pi} & \text{if } n \text{ odd.}\end{cases}
\end{aligned}
$$

for $n = 1, 2, \ldots$. The Fourier series of $f(t)$ is

$$a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt).$$

One could compute the complex Fourier series instead.

1b Look at the Fourier series of f near π. Then

$$
\begin{aligned}
\frac{\pi^2}{2} &= \frac{1}{2}\left(\lim_{t\to\pi^+} f(t) + \lim_{t\to\pi^-} f(t)\right)\\
&= a_0 + \sum_{n=1}^{\infty}(a_n\cos n\pi + b_n\sin n\pi)\\
&= a_0 + \sum_{n=1}^{\infty} a_n\cdot(-1)^n\\
&= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}(-1)^n\frac{2}{n^2}\\
&= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2\cdot(-1)^{2n}}{n^2}\\
&= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2},
\end{aligned}
$$

which gives

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\frac{\pi^2}{2} - \frac{\pi^2}{6}}{2} = \frac{\pi^2}{6}.$$

2a Setting $u(x,t) = X(x)T(t)$ gives

$$\frac{X''}{X} = \frac{T'}{c^2T} = k$$

for some constant k, so we have the two ordinary differential equations

$$X'' - kX = 0 \tag{1}$$

$$T' - c^2 kT = 0 \tag{2}$$

subject to $X(0) = 0$ and $X(1) = 0$. It is easily verified that $k \geq 0$ yields only trivial solutions (use the boundary conditions for equation (1)), so consider $k = -p^2 < 0$. The solutions of equation (1) are then of the form

$$X(x) = A_p \cos px + B_p \sin px,$$

and since $0 = X(0) = A_p$, we have

$$X(x) = B_p \sin px.$$

Since $0 = X(1) = B_p \sin p$, non-trivial solutions must have $p \in \{\pi, 2\pi, \dots\}$, giving

$$X_l(x) = B_l \sin l\pi x$$

for $l = 1, 2, \dots$. Thus equation (2) becomes

$$T_l'(t) + (c\pi l)^2 T_l(t) = 0,$$

which has solutions

$$T_l(t) = e^{-(c\pi l)^2 t}.$$

The solutions asked for are therefore linear combinations of

$$u_l(x,t) = B_l e^{-(c\pi l)^2 t} \sin l\pi x.$$

2b Letting $v(x,t) = u(x,t) - x$, we see that v satisfies the same PDE as u, i.e.

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2}.$$

The boundary condition $u(1,t) = 1$ gives $v(1,t) = 0$, so in fact v satisfies the same boundary conditions as u did in problem 2a, and hence the general solution is (from 2a)

$$v(x,t) = \sum_{l=1}^{\infty} B_l e^{-(c\pi l)^2 t} \sin l\pi x.$$

The initial condition $x(2-x) = u(x,0) = v(x,0) + x$ gives $v(x,0) = x - x^2$, and so

$$x - x^2 = v(x,t) = \sum_{l=1}^{\infty} B_l \sin l\pi x.$$

Since the left hand side of this equation defines a continuous function $f : [0,1] \to \mathbb{R}$, the Fourier series theorem gives that the B_l's are in fact the Fourier coeffieients of the 2-periodic odd extension of f. We compute these:

$$B_l = 2\int_0^1 (x - x^2) \sin l\pi x \, \mathrm{d}x = 2\int_0^1 x \sin l\pi x \, \mathrm{d}x - 2\int_0^1 x^2 \sin l\pi x \, \mathrm{d}x.$$

Calling the first term α_1 and the second term α_2, we find by partial integration

$$\alpha_1 = \frac{2}{l\pi}(-1)^{l+1}$$

$$\alpha_2 = -\alpha_1 - \frac{4}{(l\pi)^3}\left((-1)^l - 1\right).$$

Thus

$$B_l = \alpha_1 + \alpha_2 = -\frac{4}{(l\pi)^3}\left((-1)^l - 1\right) = \begin{cases} 0 & \text{if } l \text{ even} \\ \frac{8}{(l\pi)^3} & \text{if } l \text{ odd,} \end{cases}$$

and hence the solution of the PDE with given boundary conditions is

$$u(x,t) = v(x,t) + x = x + \frac{8}{\pi^3} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} e^{-(c\pi(2m+1))^2 t} \sin(2m+1)\pi x.$$

3a We have

$$y'(t) = t\big(y(t)\big)^2 + t = t\big((y(t))^2 + 1\big) = f(t, y(t)),$$

as well as $t_0 = 0$ and $y_0 = 1$. With $h = 0.2$,

$$\begin{aligned} k_1 &= hf(t_0, y_0) = 0.2 \cdot f(0,1) = 0 \\ k_2 &= hf(t_0 + h, y_0 + k_1) = 0.2 \cdot f(0.2, 1) = 0.08. \end{aligned}$$

Thus

$$y(0.2) \approx \hat{y} = y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1.04.$$

3b From problem 3a and the text we have

$$y(0.2) = \hat{y} + \hat{\varepsilon} = 1.04 + \hat{\varepsilon}.$$

With *two* steps, we have $y(0.2) \approx \tilde{y}$, i.e.

$$y(0.2) = \tilde{y} + 2\tilde{\varepsilon} = 1.040714 + 2\tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ is the error in each of the two steps. Since the local error in Heun's method is of order 3, and $\tilde{y}$ is computed with half the step size of $\hat{y}$, we have

$$\tilde{\varepsilon} \approx \frac{1}{2^3}\hat{\varepsilon} = \frac{1}{8}\hat{\varepsilon}.$$

Thus

$$1.04 + \hat{\varepsilon} = y(0.2) = 1.040714 + 2\tilde{\varepsilon} \approx 1.040714 + 2\frac{1}{8}\hat{\varepsilon},$$

which when solved for $\hat{\varepsilon}$ results in $\hat{\varepsilon} \approx 0.96 \cdot 10^{-3}$.

4a We compute

$$\begin{aligned} \mathscr{F}(f)(\omega) = \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}e^{-i\omega x}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(a+i\omega)x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}(a+i\omega)}. \end{aligned}$$

From now on, set $a = 1$ to get

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}(1+i\omega)}.$$

Observe that

$$\begin{aligned} 0 = f(-1) = \mathscr{F}^{-1}(\hat{f})(-1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(1+i\omega)} e^{i\omega\cdot(-1)}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+i\omega}(\cos\omega - i\sin\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-i\omega}{1+\omega^2}(\cos\omega - i\sin\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega - i\sin\omega - i\omega\cos\omega - \omega\sin\omega}{1+\omega^2}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega - \omega\sin\omega}{1+\omega^2}\,\mathrm{d}\omega - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\sin\omega + \omega\cos\omega}{1+\omega^2}\,\mathrm{d}\omega. \end{aligned}$$

The second integral on the last line must be zero since the left hand side is real (in fact, the integrand is odd, so the integral is indeed zero). The integrand in the first integral is even, so

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega - \omega\sin\omega}{1+\omega^2}\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^{\infty} \frac{\cos\omega - \omega\sin\omega}{1+\omega^2}\,\mathrm{d}\omega,$$

and so the integral we were asked to compute is 0.

4b For $b > 0$, define $g_b : \mathbb{R} \to \mathbb{R}$ by $g_b(x) = e^{-bx^2}$. We recognize the integral as a convolution, so that the equation in the problem becomes

$$f * g_2 = g_1.$$

Fourier transforming gives

$$\sqrt{2\pi}\hat{f} \cdot \hat{g}_2 = \hat{g}_1,$$

and we know well that

$$\hat{g}_b(\omega) = \frac{1}{\sqrt{2b}} e^{-\omega^2/4b},$$

so we have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{\sqrt{2}} e^{-\omega^2/4}}{\frac{1}{\sqrt{4}} e^{-\omega^2/8}} = \frac{1}{\sqrt{\pi}} e^{-\omega^2/8} = \frac{2}{\sqrt{\pi}} \hat{g}_2(\omega).$$

Taking the inverse Fourier transform gives

$$f(x) = \frac{2}{\sqrt{\pi}} g_2(x) = \frac{2}{\sqrt{\pi}} e^{-2x^2}.$$

5a While it is perfectly OK to approach the problem directly using Newton's divided differences or a Lagrange scheme, we can save ourselves some work by noticing that the polynomial q given by $q(x) = p(x) - 1$ has zeros at 0 and 1. We thus have

$$p(x) - 1 = q(x) = x(x-1)r(x)$$

for some polynomial r. In particular, the unused interpolation data becomes

$$\begin{aligned} 7 = -2 \cdot (-2-1)r(-2) + 1 = 6r(-2) + 1 &\implies r(-2) = 1 \\ 3 = -1 \cdot (-1-1)r(-1) + 1 = 2r(-1) + 1 &\implies r(-1) = 1 \\ 3 = 2 \cdot (2-1)r(2) + 1 = 2r(2) + 1 &\implies r(2) = 1. \end{aligned}$$

It is clear, then, that $r(x) = 1$. Thus,

$$p(x) = q(x) + 1 = x(x-1) \cdot 1 + 1 = 1 - x + x^2.$$

5b Simpson's method is exact for polynomials of degree less than or equal to 3, and will thus give the exact value for the integral of p: With $n = 4$ ($h = 1$) we find

$$\int_{-2}^{2} p(x)\,\mathrm{d}x = \frac{1}{3}(7 + 4 \cdot 3 + 2 \cdot 1 + 4 \cdot 1 + 3) = \frac{28}{3}.$$